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Abstract. A simple precessing ellipse suitable for satellites with moderate eccentricities and inclinations is built in the Hamiltonian context of the J_2 problem. Easy to use as a first intermediary orbit, it provides a substantially closer starting point for series expansions than the regular Keplerian ellipse.

Key words. planets and satellites: general – celestial mechanics

1. Introduction

The J_2 problem deals with the motion of a satellite around an oblate planet, and an intermediary orbit is an integrable approximation used as a starting point for developing the problem more fully. Using the spherical coordinates $r, \theta, \varphi, P_r, P_\theta$, and P_φ defined in the planet's equatorial plane and defining the functions

$$\sigma = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}, \quad \sigma_x = P_\varphi \sin\theta - P_\theta \tan\varphi \cos\theta,$$

$$\sigma_y = -P_\varphi \cos\theta - P_\theta \tan\varphi \sin\theta, \quad \sigma_z = P_\theta,$$

the Hamiltonian of the J_2 problem is

$$K = \frac{1}{2}P_r^2 + \frac{1}{2}\left(\frac{\sigma^2}{r^2} - \frac{2\mu}{r}\right) - \frac{\mu J}{r^3} + \frac{3\mu J}{r^3}\sin^2\varphi,$$

where μ is the mass of the planet multiplied by the gravitational constant, and $J = \frac{1}{2}J_2 r_e^2$ stands for half the planet's first zonal harmonic coefficient J_2 multiplied by the square of its equatorial radius.

This is not an integrable problem (Irigoyen & Simó 1993), but it can be approximated by series expansions from suitable intermediary orbits (Deprit 1981; Floría 1993). For satellites with moderate eccentricities and inclinations, intermediary orbits can be less sophisticated than the aforementioned ones. With

$$K_0 = \frac{1}{2}P_r^2 + \frac{1}{2}\left(\frac{\sigma^2}{r^2} - \frac{2\mu}{r}\right), \quad \widetilde{J} = J\left(\frac{3}{2}\frac{\sigma_z^2}{\sigma^2} - \frac{1}{2}\right),$$

the previous Hamiltonian reads

$$K = K_0 - \frac{\mu\widetilde{J}}{r^3} - \frac{3\mu J}{r^3}\left(\frac{\sigma_x^2 + \sigma_y^2}{2\sigma^2}\right)\left(1 - 2\frac{\sigma^2\sin^2\varphi}{\sigma_x^2 + \sigma_y^2}\right),$$

where the first term following K_0 takes into account the satellite's inclination, and the second term is mainly a short-period oscillating perturbation. At first sight, there are then two handy candidates for a simple intermediary orbit, deriving from the Hamiltonian K_0 and the Hamiltonian

$$K_1 = \frac{1}{2}P_r^2 + \frac{1}{2}\left(\frac{\sigma^2}{r^2} - \frac{2\mu}{r}\right) - \frac{\mu\widetilde{J}}{r^3}.$$

Both are integrable problems. The first one (the 2-Body problem) is easy to solve but leads to rather long series expansions to reach K. The second one provides a much closer starting point but involves the somewhat less convenient elliptic functions in the solution. The purpose of this paper is to build an easy-to-use intermediary orbit between K_0 and K_1 suitable for satellites with moderate eccentricities and inclinations.

2. K_0 *and the Keplerian ellipse*

Some classical results are useful for the following sections. The orbit deriving from K_0 can be defined by the osculating elements computed from the prime integrals of the motion $\sigma_x, \sigma_y, \sigma_z$, and $K_0 = k_0$ (Duriez 1989). When $-\mu^2/2\sigma^2 < k_0 < 0$, there are two different positive values of r for which $P_r = 0$. The motion is bounded and the orbit is the *Keplerian ellipse* defined by a, e, i, Ω, ω, and τ:

$$a = -\frac{\mu}{2k_0}, \quad e = \sqrt{1 + 2k_0\frac{\sigma^2}{\mu^2}}, \quad \sin i = \frac{\sqrt{\sigma_x^2 + \sigma_y^2}}{\sigma},$$

$$\cos i = \frac{\sigma_z}{\sigma}, \quad \sin\Omega = \frac{\sigma_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}, \quad \cos\Omega = \frac{-\sigma_y}{\sqrt{\sigma_x^2 + \sigma_y^2}}$$

if $\sin i \neq 0$. The value of τ is computed from the initial values of the canonical variables at $t = t_0$:

$$e\sin u = \frac{rP_r}{\sqrt{\mu a}}, \quad e\cos u = 1 - \frac{r}{a},$$

$$\tau = (e\sin u - u)\sqrt{\frac{a^3}{\mu}} + t_0,$$

and the value of ω can be computed from

$$\sin v = \frac{\sqrt{1-e^2}\sin u}{1-e\cos u}, \quad \cos v = \frac{\cos u - e}{1-e\cos u};$$

$\sin i = 0: \quad v+\omega = \theta; \quad \sin i \neq 0:$

$$\sin(v+\omega) = \frac{\sigma\sin\varphi}{\sqrt{\sigma_x^2+\sigma_y^2}}, \quad \cos(v+\omega) = \frac{P_\varphi\cos\varphi}{\sqrt{\sigma_x^2+\sigma_y^2}}.$$

The variations of r, θ, and φ as functions of the time are

$$u - e\sin u = \sqrt{\frac{\mu}{a^3}}(t-\tau), \quad r = a(1-e\cos u),$$

$$\sin\varphi = \sin i \sin(v+\omega),$$

$$\cos\varphi\sin\theta = \cos(v+\omega)\sin\Omega + \sin(v+\omega)\cos\Omega\cos i,$$

$$\cos\varphi\cos\theta = \cos(v+\omega)\cos\Omega - \sin(v+\omega)\sin\Omega\cos i.$$

The canonical Delaunay transformation

$$\ell = \sqrt{\frac{\mu}{a^3}}(t-\tau), \quad L = \sqrt{\mu a},$$

$$g = \omega, \quad G = \sqrt{\mu a(1-e^2)} = \sigma,$$

$$h = \Omega, \quad H = \sqrt{\mu a(1-e^2)}\cos i = \sigma_z$$

turns the Hamiltonian of the J_2 problem into

$$K = -\frac{\mu^2}{2L^2} - \frac{\mu\widetilde{J}}{r^3} - \frac{3\mu J}{r^3}\frac{\sin^2 i}{2}\cos 2(v+\omega),$$

where $\widetilde{J} = J\left(1-\frac{3}{2}\sin^2 i\right)$, and where non-canonical variables have to be expressed in canonical ones.

3. The dressing of K_1

The functions σ_x and σ_y are no longer prime integrals of the differential system derived from K_1, but $\sigma_x^2+\sigma_y^2$ and σ_z are still prime integrals of this system, together with $K_1 = k_1$. Finding different real values of r for which $P_r = 0$ requires

$$108\mu^2\widetilde{J}^2k_1^2 + 2\sigma^2\left(18\mu^2\widetilde{J}-\sigma^4\right)k_1 + \mu^2\left(16\mu^2\widetilde{J}-\sigma^4\right) < 0,$$

leading to

$$\frac{12\mu^2\widetilde{J}}{\sigma^4} < 1, \quad R = \sqrt{1-\frac{12\mu^2\widetilde{J}}{\sigma^4}}, \quad D^3 = 2\mu\widetilde{J},$$

$$\frac{\sigma^6\left(-1+3R^2-2R^3\right)}{54D^6} < k_1 < \frac{\sigma^6\left(-1+3R^2+2R^3\right)}{54D^6}.$$

The lower bound of k_1 is always negative. Raising the energy level of a Hamiltonian in order that the lower bound of the motion range equals 0 generally simplifies the Hamiltonian's expression:

$$K_0 + \frac{\mu^2}{2\sigma^2} = \frac{1}{2}P_r^2 + \frac{1}{2}\left(\frac{\mu}{\sigma} - \frac{\sigma}{r}\right)^2.$$

Let's try the process on K_1:

$$\begin{aligned}
2 \times &\left[K_1 - \frac{1}{2}P_r^2 + \frac{\sigma^6\left(1-3R^2+2R^3\right)}{54D^6}\right] \\
&= \frac{\sigma^6\left(1-3R^2+2R^3\right)}{27D^6} - \frac{2\mu}{r} + \frac{\sigma^2}{r^2} - \frac{2\mu\widetilde{J}}{r^3} \\
&= \frac{\sigma^6}{27D^6} - \frac{\sigma^4}{3D^3r} + \frac{\sigma^2}{r^2} - \frac{D^3}{r^3} \\
&\quad -\frac{3\sigma^6R^2}{27D^6}\left(1-\frac{2R}{3}\right) + \frac{\sigma^4}{3D^3r}\left(1-\frac{12\mu^2\widetilde{J}}{\sigma^4}\right) \\
&= \left(\frac{\sigma^2}{3D^2} - \frac{D}{r}\right)^3 \\
&\quad -3\left(\frac{\sigma^2}{3D^2} - \frac{D}{r}\right)\left(\frac{\sigma^2R}{3D^2}\right)^2 + 2\left(\frac{\sigma^2R}{3D^2}\right)^3 \\
&= \left[\frac{\sigma^2(1+2R)}{3D^2} - \frac{D}{r}\right]\left[\frac{\sigma^2(1-R)}{3D^2} - \frac{D}{r}\right]^2 \\
&= \left(\frac{1+2R}{3} - \frac{2\mu\widetilde{J}}{\sigma^2r}\right)\left[\frac{\mu}{\sigma}\left(\frac{2}{1+R}\right) - \frac{\sigma}{r}\right]^2 \\
&= \left(\frac{1+2R}{3}\right)\left[\frac{\sigma^2}{r^2} - \frac{2\mu}{r}\left(\frac{2}{1+R}\right)\right] - \frac{2\mu\widetilde{J}}{r^3} \\
&\quad \times\left[\frac{\mu r}{\sigma^2}\left(\frac{2}{1+R}\right) - 1\right]^2 + \frac{\mu^2}{\sigma^2}\left(\frac{1+2R}{3}\right)\left(\frac{2}{1+R}\right)^2.
\end{aligned}$$

With the functions

$$\widetilde{\sigma} = \sigma\sqrt{\frac{1+2R}{3}}, \quad \widetilde{\mu} = \frac{2}{3}\mu\left(\frac{1+2R}{1+R}\right),$$

and the relation

$$\frac{\sigma^6\left(1-3R^2+2R^3\right)}{54D^6} = \frac{\mu^2}{2\sigma^2}\left(\frac{1+2R}{3}\right)\left(\frac{2}{1+R}\right)^2,$$

the Hamiltonian's expression is

$$K_1 = \frac{1}{2}P_r^2 + \frac{1}{2}\left(\frac{\widetilde{\sigma}^2}{r^2} - \frac{2\widetilde{\mu}}{r}\right) - \frac{\mu\widetilde{J}}{r^3}\left(\frac{\widetilde{\mu}r}{\widetilde{\sigma}^2} - 1\right)^2.$$

4. $\widetilde{K}_0$ and the Hamiltonian ellipse

Let's define the Hamiltonian

$$\widetilde{K}_0 = \frac{1}{2}P_r^2 + \frac{1}{2}\left(\frac{\widetilde{\sigma}^2}{r^2} - \frac{2\widetilde{\mu}}{r}\right) = \widetilde{k}_0.$$

The first step for finding the solution is similar to solving the 2-Body problem. When $-\widetilde{\mu}^2/2\widetilde{\sigma}^2 < \widetilde{k}_0 < 0$, the motion is bounded:

$$a = -\frac{\widetilde{\mu}}{2\widetilde{k}_0}, \quad e = \sqrt{1+2\widetilde{k}_0\frac{\widetilde{\sigma}^2}{\widetilde{\mu}^2}},$$

$$a(1-e) \le r \le a(1+e).$$

The canonical *action*-variable (Henrard 1989)

$$I = \frac{1}{2\pi}\oint P_r\,\mathrm{d}r$$
$$= \frac{1}{\pi}\int_{a(1-e)}^{a(1+e)}\left(2\widetilde{k}_0 - \frac{\widetilde{\sigma}^2}{r^2} + \frac{2\widetilde{\mu}}{r}\right)^{\frac{1}{2}}\mathrm{d}r = \frac{\widetilde{\mu}}{\sqrt{-2\widetilde{k}_0}} - \widetilde{\sigma}$$

turns the Hamiltonian into

$$\widetilde{K}_0 = -\frac{\widetilde{\mu}^2}{2\,(I+\widetilde{\sigma})^2},$$

and the computation of the canonical *angle*-variable

$$\psi = \frac{\partial\widetilde{K}_0}{\partial I}\int_{a(1-e)}^{r}\frac{\partial P_r}{\partial\widetilde{k}_0}\,\mathrm{d}\rho$$
$$= \frac{\widetilde{\mu}^2}{(I+\widetilde{\sigma})^3}\int_{a(1-e)}^{r}\left(2\widetilde{k}_0 - \frac{\widetilde{\sigma}^2}{\rho^2} + \frac{2\widetilde{\mu}}{\rho}\right)^{-\frac{1}{2}}\mathrm{d}\rho$$
$$= \arccos\left(\frac{a-r}{ae}\right) - e\sin\left[\arccos\left(\frac{a-r}{ae}\right)\right]$$

leads to

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\partial\widetilde{K}_0}{\partial I} = \sqrt{\frac{\widetilde{\mu}}{a^3}},$$

$$r = a\,(1-e\cos u)\,,\quad u - e\sin u = \sqrt{\frac{\widetilde{\mu}}{a^3}}\,(t-\tau)\,,$$

where the value of τ is computed from the initial values of the canonical variables at $t = t_0$:

$$e\sin u = \frac{rP_r}{\sqrt{\widetilde{\mu}a}},\quad e\cos u = 1 - \frac{r}{a},$$

$$\tau = (e\sin u - u)\sqrt{\frac{a^3}{\widetilde{\mu}}} + t_0.$$

The differential system derived from $\widetilde{K}_0$ is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = P_r,\quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{C_1P_\theta}{r^2\cos^2\varphi} + \frac{C_2}{r^2},\quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{C_1P_\varphi}{r^2},$$

$$\frac{\mathrm{d}P_r}{\mathrm{d}t} = \frac{\widetilde{\sigma}^2}{r^3} - \frac{\widetilde{\mu}}{r^2},\quad \frac{\mathrm{d}P_\theta}{\mathrm{d}t} = 0,\quad \frac{\mathrm{d}P_\varphi}{\mathrm{d}t} = -\frac{C_1P_\theta^2\sin\varphi}{r^2\cos^3\varphi},$$

where

$$C_1 = \frac{\widetilde{\sigma}}{\sigma}\frac{\partial\widetilde{\sigma}}{\partial\sigma} - \frac{r}{\sigma}\frac{\partial\widetilde{\mu}}{\partial\sigma},\quad C_2 = \widetilde{\sigma}\frac{\partial\widetilde{\sigma}}{\partial\sigma_z} - r\frac{\partial\widetilde{\mu}}{\partial\sigma_z},$$

$$\frac{\partial R}{\partial\sigma} = \frac{12\mu^2J}{R\,\sigma^5}\left(\frac{9}{2}\frac{\sigma_z^2}{\sigma^2} - 1\right),\quad \frac{\partial R}{\partial\sigma_z} = -\frac{18\mu^2J\sigma_z}{R\,\sigma^6},$$

$$\frac{\partial\widetilde{\sigma}}{\partial\sigma} = \left(\frac{1+2R}{3} + \frac{\sigma}{3}\frac{\partial R}{\partial\sigma}\right)\sqrt{\frac{3}{1+2R}},$$

$$\frac{\partial\widetilde{\sigma}}{\partial\sigma_z} = \frac{\sigma}{3}\frac{\partial R}{\partial\sigma_z}\sqrt{\frac{3}{1+2R}},$$

$$\frac{\partial\widetilde{\mu}}{\partial\sigma} = \frac{2}{3}\frac{\partial R}{\partial\sigma}\frac{\mu}{(1+R)^2},\quad \frac{\partial\widetilde{\mu}}{\partial\sigma_z} = \frac{2}{3}\frac{\partial R}{\partial\sigma_z}\frac{\mu}{(1+R)^2}.$$

Defining the inclination by

$$\sin i = \frac{\sqrt{\sigma_x^2+\sigma_y^2}}{\sigma},\quad \cos i = \frac{\sigma_z}{\sigma},$$

and using the relation

$$\mathrm{d}t = \sqrt{\frac{a^3}{\widetilde{\mu}}}\,(1-e\cos u)\,\mathrm{d}u$$

turn the equation for φ into

$$\frac{\cos\varphi}{\sqrt{\sin^2 i - \sin^2\varphi}}\,\mathrm{d}\varphi = \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma}\frac{\sqrt{1-e^2}}{1-e\cos u} - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma}\right)\mathrm{d}u.$$

The variable φ is not periodic. Let's use the relation

$$\arcsin\left(\frac{\sin\varphi}{\sin i}\right) = \arctan\left(\frac{\sigma\sin\varphi}{P_\varphi\cos\varphi}\right),$$

and extend from $-\pi$ to π the value range of the right-hand function according to the signs of $\sin\varphi$ and P_φ. Then, if m is the largest integer $\leq \sqrt{\widetilde{\mu}/a^3}\,(t-\tau)\,/2\pi$, n is a suitable integer, and v is defined by

$$\sin v = \frac{\sqrt{1-e^2}\sin u}{1-e\cos u},\quad \cos v = \frac{\cos u - e}{1-e\cos u},$$

where $0 \leq v < 2\pi$, the solution is

$$2n\pi + \arctan\left(\frac{\sigma\sin\varphi}{P_\varphi\cos\varphi}\right)$$
$$= \frac{\partial\widetilde{\sigma}}{\partial\sigma}(2m\pi + v) - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma}u + \omega$$
$$= v + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma} - 1\right)(2m\pi + v) - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma}u + \omega + 2m\pi$$
$$= v + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma} - 1\right)(2m\pi + v - u)$$
$$+\left(\frac{\partial\widetilde{\sigma}}{\partial\sigma} - 1 - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma}\right)e\sin u$$
$$+\omega + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma} - 1 - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma}\right)\sqrt{\frac{\widetilde{\mu}}{a^3}}\,(t-\tau) + 2m\pi.$$

The variable r is periodic of period $2\pi\sqrt{a^3/\widetilde{\mu}}$. The value of u can then be computed from

$$u - e\sin u = \sqrt{\frac{\widetilde{\mu}}{a^3}}\,(t-\tau) - 2m\pi,$$

which means that $0 \leq u < 2\pi$. This convention turns u into $2m\pi + u$ in the previous solution. Thus, defining

$$\widetilde{v} = v + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma} - 1\right)(v-u) + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma} - 1 - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma}\right)e\sin u,$$

$$\widetilde{\omega} = \omega + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma} - 1 - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma}\right)\sqrt{\frac{\widetilde{\mu}}{a^3}}\,(t-\tau)\,,$$

where $0 \le u, v < 2\pi$, the value of ω can be computed from

$\sin i = 0$: $\quad \widetilde{v} + \widetilde{\omega} = \theta$; $\quad \sin i \neq 0$:

$$\sin(\widetilde{v} + \widetilde{\omega}) = \frac{\sigma \sin\varphi}{\sqrt{\sigma_x^2 + \sigma_y^2}}, \quad \cos(\widetilde{v} + \widetilde{\omega}) = \frac{P_\varphi \cos\varphi}{\sqrt{\sigma_x^2 + \sigma_y^2}},$$

and the variations of φ are given by

$$\sin\varphi = \sin i \sin(\widetilde{v} + \widetilde{\omega}).$$

The angle $\widetilde{v}$ periodically oscillates around v, and $\widetilde{\omega}$ is a precessing angle. They generalize v and ω when $J \neq 0$. The variations of θ involve them, too. From the differential system derived from $\widetilde{K}_0$, the equation for θ is

$$\begin{aligned} \mathrm{d}\theta &= \frac{\cos i}{\cos\varphi \sqrt{\sin^2 i - \sin^2\varphi}} \mathrm{d}\varphi \\ &+ \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma_z} \frac{\sqrt{1-e^2}}{1 - e\cos u} - \sqrt{\frac{a}{\widetilde{\mu}}} \frac{\partial\widetilde{\mu}}{\partial\sigma_z} \right) \mathrm{d}u. \end{aligned}$$

The variable θ is not periodic. Let's use the relation

$$\arcsin\left(\frac{\tan\varphi}{\tan i}\right) = \arctan\left(\frac{\sigma_z \sin\varphi}{P_\varphi \cos\varphi}\right),$$

and extend from $-\pi$ to π the value range of the right-hand function according to the signs of $\sin\varphi$ and P_φ. Then, if n is a suitable integer, and $0 \le u, v < 2\pi$, the solution is

$$\begin{aligned} \theta &= 2n\pi + \arctan\left(\frac{\sigma_z \sin\varphi}{P_\varphi \cos\varphi}\right) \\ &+ \frac{\partial\widetilde{\sigma}}{\partial\sigma_z}(v - u) + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma_z} - \sqrt{\frac{a}{\widetilde{\mu}}} \frac{\partial\widetilde{\mu}}{\partial\sigma_z} \right) e \sin u \\ &+ \Omega + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma_z} - \sqrt{\frac{a}{\widetilde{\mu}}} \frac{\partial\widetilde{\mu}}{\partial\sigma_z} \right) \sqrt{\frac{\widetilde{\mu}}{a^3}} (t - \tau). \end{aligned}$$

Thus, defining

$$\widetilde{\theta} = \theta - \frac{\partial\widetilde{\sigma}}{\partial\sigma_z}(v - u) - \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma_z} - \sqrt{\frac{a}{\widetilde{\mu}}} \frac{\partial\widetilde{\mu}}{\partial\sigma_z} \right) e \sin u,$$

$$\widetilde{\Omega} = \Omega + \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma_z} - \sqrt{\frac{a}{\widetilde{\mu}}} \frac{\partial\widetilde{\mu}}{\partial\sigma_z} \right) \sqrt{\frac{\widetilde{\mu}}{a^3}} (t - \tau),$$

the value of Ω can be computed from

$$\sin\left(\widetilde{\theta} - \widetilde{\Omega}\right) = \frac{\sigma_z \tan\varphi}{\sqrt{\sigma_x^2 + \sigma_y^2}}, \quad \cos\left(\widetilde{\theta} - \widetilde{\Omega}\right) = \frac{P_\varphi}{\sqrt{\sigma_x^2 + \sigma_y^2}}$$

if $\sin i \neq 0$, and the variations of θ come from

$$\cos\varphi \sin\widetilde{\theta} = \cos(\widetilde{v} + \widetilde{\omega}) \sin\widetilde{\Omega} + \sin(\widetilde{v} + \widetilde{\omega}) \cos\widetilde{\Omega} \cos i,$$

$$\cos\varphi \cos\widetilde{\theta} = \cos(\widetilde{v} + \widetilde{\omega}) \cos\widetilde{\Omega} - \sin(\widetilde{v} + \widetilde{\omega}) \sin\widetilde{\Omega} \cos i.$$

The angle $\widetilde{\theta}$ periodically oscillates around θ, and $\widetilde{\Omega}$ is a precessing angle. They generalize θ and Ω when $J \neq 0$. For initial values matching $\sigma^6 - 6\mu^2 J \left(3\sigma_z^2 - \sigma^2\right) > 0$ and $-\widetilde{\mu}^2/2\widetilde{\sigma}^2 < \widetilde{k}_0 < 0$, the solution of $\widetilde{K}_0$ is a precessing ellipse with the same fixed inclination as the Keplerian ellipse. It can be described from the osculating elements a, e, i, Ω, ω, and τ defining what can be called the *Hamiltonian ellipse* related to the J_2 problem.

5. Canonical variables

The expression of $\widetilde{K}_0$ as a function of I and ψ implies

$$\frac{\partial\widetilde{K}_0}{\partial I} = \frac{\mathrm{d}\psi}{\mathrm{d}t}, \quad \frac{\partial\widetilde{K}_0}{\partial\sigma} = \frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\omega} + \psi), \quad \frac{\partial\widetilde{K}_0}{\partial\sigma_z} = \frac{\mathrm{d}\widetilde{\Omega}}{\mathrm{d}t}.$$

The transformation

$$\ell_2 = \psi = \sqrt{\frac{\widetilde{\mu}}{a^3}}(t - \tau), \quad L_2 = I + \sigma = \sqrt{\widetilde{\mu} a} - \widetilde{\sigma} + \sigma,$$

$$g_2 = (\widetilde{\omega} + \psi) - \psi = \widetilde{\omega}, \quad G_2 = \sigma, \quad h_2 = \widetilde{\Omega}, \quad H_2 = \sigma_z$$

is then canonical and turns the Hamiltonian into

$$\widetilde{K}_0 = -\frac{\widetilde{\mu}^2}{2(L_2 - G_2 + \widetilde{\sigma})^2}.$$

In order to apply the Lie algorithm to a Hamiltonian (Deprit 1969), it is useful to reduce the number of variables in the intermediary orbit. The canonical transformation

$$\ell_1 = \ell_2, \quad L_1 = L_2 - G_2 + \widetilde{\sigma} = \sqrt{\widetilde{\mu} a},$$

$$\begin{aligned} g_1 &= g_2 - \left(\frac{\partial\widetilde{\sigma}}{\partial G_2} - 1\right)\ell_2 = \widetilde{\omega} - \left(\frac{\partial\widetilde{\sigma}}{\partial\sigma} - 1\right)\sqrt{\frac{\widetilde{\mu}}{a^3}}(t - \tau) \\ &= \omega - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma}\sqrt{\frac{\widetilde{\mu}}{a^3}}(t - \tau), \quad G_1 = G_2, \end{aligned}$$

$$\begin{aligned} h_1 &= h_2 - \frac{\partial\widetilde{\sigma}}{\partial H_2}\ell_2 = \widetilde{\Omega} - \frac{\partial\widetilde{\sigma}}{\partial\sigma_z}\sqrt{\frac{\widetilde{\mu}}{a^3}}(t - \tau) \\ &= \Omega - \sqrt{\frac{a}{\widetilde{\mu}}}\frac{\partial\widetilde{\mu}}{\partial\sigma_z}\sqrt{\frac{\widetilde{\mu}}{a^3}}(t - \tau), \quad H_1 = H_2 \end{aligned}$$

begins the process:

$$\widetilde{K}_0 = -\frac{\widetilde{\mu}^2}{2L_1^2}.$$

The Hamiltonian still depends on G_1 and H_1 because of $\widetilde{\mu}$. The canonical transformation

$$\ell = \frac{\widetilde{\mu}}{\mu}\ell_1, \quad L = \frac{\mu}{\widetilde{\mu}}L_1,$$

$$g = g_1 + \frac{L_1}{\widetilde{\mu}}\frac{\partial\widetilde{\mu}}{\partial G_1}\ell_1 = \omega, \quad G = G_1,$$

$$h = h_1 + \frac{L_1}{\widetilde{\mu}}\frac{\partial\widetilde{\mu}}{\partial H_1}\ell_1 = \Omega, \quad H = H_1$$

ends the process:

$$\widetilde{K}_0 = -\frac{\mu^2}{2L^2}.$$

Then, with the relation

$$E = \frac{\widetilde{\mu} r}{\widetilde{\sigma}^2} - 1 = \left(\frac{e}{1 - e}\right)\left(\frac{e - \cos u}{1 + e}\right),$$

Table 1. Standard deviations for r, θ, and φ over one period of r.

J	e	e_k	e_h	Keplerian ellipse: s_r	s_θ	s_φ	Hamiltonian ellipse: s_r	s_θ	s_φ
10^{-5}	0.1	0.10	0.10	5.21×10^{-5}	7.73×10^{-4}	1.54×10^{-4}	5.21×10^{-6}	2.33×10^{-5}	6.11×10^{-6}
	0.3	0.30	0.30	3.88×10^{-5}	7.42×10^{-4}	1.52×10^{-4}	1.53×10^{-5}	8.72×10^{-5}	6.65×10^{-6}
	0.5	0.50	0.50	3.36×10^{-5}	9.53×10^{-4}	1.84×10^{-4}	2.42×10^{-5}	2.94×10^{-4}	3.29×10^{-5}
10^{-4}	0.1	0.10	0.10	5.21×10^{-4}	7.74×10^{-3}	1.54×10^{-3}	5.26×10^{-5}	2.35×10^{-4}	6.15×10^{-5}
	0.3	0.30	0.30	3.88×10^{-4}	7.44×10^{-3}	1.51×10^{-3}	1.54×10^{-4}	8.81×10^{-4}	6.72×10^{-5}
	0.5	0.50	0.50	3.36×10^{-4}	9.58×10^{-3}	1.84×10^{-3}	2.43×10^{-4}	2.97×10^{-3}	3.32×10^{-4}
10^{-3}	0.1	0.10	0.11	5.20×10^{-3}	7.88×10^{-2}	1.55×10^{-2}	5.82×10^{-4}	2.56×10^{-3}	6.48×10^{-4}
	0.3	0.30	0.31	3.90×10^{-3}	7.64×10^{-2}	1.51×10^{-2}	1.61×10^{-3}	9.70×10^{-3}	8.01×10^{-4}
	0.5	0.50	0.51	3.38×10^{-3}	1.00×10^{-1}	1.85×10^{-2}	2.51×10^{-3}	3.34×10^{-2}	4.01×10^{-3}
10^{-2}	0.1	0.10	0.22	5.17×10^{-2}	9.91×10^{-1}	1.79×10^{-1}	1.34×10^{-2}	8.80×10^{-2}	1.62×10^{-2}
	0.3	0.30	0.41	4.11×10^{-2}	1.09	2.07×10^{-1}	2.56×10^{-2}	0.32	6.55×10^{-2}
	0.5	0.50	0.64	4.46×10^{-2}	3.04	1.21×10^{-1}	5.29×10^{-2}	3.00	1.16×10^{-1}

Table 2. Standard deviations for r, θ, and φ over five periods of r.

J	e	e_k	e_h	Keplerian ellipse: s_r	s_θ	s_φ	Hamiltonian ellipse: s_r	s_θ	s_φ
10^{-5}	0.1	0.10	0.10	3.84×10^{-5}	1.49×10^{-4}	1.53×10^{-4}	1.23×10^{-5}	5.06×10^{-5}	6.06×10^{-6}
	0.3	0.30	0.30	1.11×10^{-4}	5.09×10^{-4}	2.04×10^{-4}	4.33×10^{-5}	2.04×10^{-4}	2.79×10^{-5}
	0.5	0.50	0.50	1.97×10^{-4}	1.19×10^{-3}	3.45×10^{-4}	1.02×10^{-4}	6.43×10^{-4}	1.00×10^{-4}
10^{-4}	0.1	0.10	0.10	3.87×10^{-4}	1.51×10^{-3}	1.52×10^{-3}	1.24×10^{-4}	5.13×10^{-4}	6.20×10^{-5}
	0.3	0.30	0.30	1.11×10^{-3}	5.12×10^{-3}	2.05×10^{-3}	4.35×10^{-4}	2.06×10^{-3}	2.84×10^{-4}
	0.5	0.50	0.50	1.97×10^{-3}	1.19×10^{-2}	3.51×10^{-3}	1.02×10^{-3}	6.50×10^{-3}	1.04×10^{-3}
10^{-3}	0.1	0.11	0.11	4.23×10^{-3}	1.67×10^{-2}	1.54×10^{-2}	1.38×10^{-3}	5.82×10^{-3}	8.40×10^{-4}
	0.3	0.31	0.31	1.15×10^{-2}	5.34×10^{-2}	2.25×10^{-2}	4.58×10^{-3}	2.22×10^{-2}	3.93×10^{-3}
	0.5	0.52	0.51	2.00×10^{-2}	1.24×10^{-1}	4.53×10^{-2}	1.07×10^{-2}	7.09×10^{-2}	1.70×10^{-2}
10^{-2}	0.1	0.14	0.19	1.03×10^{-1}	4.58	2.40×10^{-1}	3.20×10^{-2}	0.20	5.02×10^{-2}
	0.3	0.29	0.32	7.83×10^{-2}	4.34	0.48	2.92×10^{-2}	0.10	0.46
	0.5	0.39	0.64	0.11	13.79	0.28	0.21	14.00	0.19

the Hamiltonian of the J_2 problem turns into

$$K = -\frac{\mu^2}{2L^2} - \frac{\mu\widetilde{J}}{r^3} E^2 - \frac{3\mu J}{r^3} \frac{\sin^2 i}{2} \cos 2(\widetilde{\upsilon} + \widetilde{\omega}),$$

where $\widetilde{J} = J\left(1 - \frac{3}{2}\sin^2 i\right)$, and where non-canonical variables have to be expressed in canonical ones. Building an improved intermediary orbit requires that the second term of K be smaller than the one obtained from the classical process. Then, with $|e - \cos u| \leq 1 + e$, $\widetilde{K}_0$ offers a better intermediary orbit than K_0 when $e < \frac{1}{2}$. And the smaller the better…

6. Numerical simulations

Two sets of numerical simulations are performed in dimensionless units with four different values of J and $\mu = 1$. In this case, the value of J would be around 1.23×10^{-5} and 7.96×10^{-4} for the Earth and Saturn, respectively. The simulations start with initial values computed from $a = 0.5$, three different values of e $(0.1, 0.3, 0.5)$, and $i = 0.2$, where a, e, and i are the osculating elements of the J_2 problem at the starting time. The first set of simulations compares K_0 and $\widetilde{K}_0$ in their roles as first intermediary orbits to be used as a starting point for developments. The discrepancies between each approximation and the J_2 problem are computed for r, θ, and φ during one period of the variable r of the considered approximation. The actual eccentricities e_k (Keplerian ellipse) and e_h (Hamiltonian ellipse) and the standard deviations s_r, s_θ, and s_φ for each approximation are summarized in Table 1. The second set of simulations compares K_0 and $\widetilde{K}_0$ in their roles as approximate solutions to be used for predictions over short periods of time. The discrepancies between each approximation and the J_2 problem are computed for r, θ, and φ during five periods of the variable r of the considered approximation, but the constants of each approximation are fitted by least squares to the J_2 problem. The fitted eccentricities e_k and e_h and the standard deviations s_r, s_θ, and s_φ for each approximation are summarized in Table 2. In both cases, $\widetilde{K}_0$ is a better option for moderate eccentricities, leading up to ten times smaller values, and a rather worse option for $e_h > \frac{1}{2}$, as expected from the theory.

7. Perturbation

The solution of K is obtained from the perturbation of the canonical Delaunay-like variables ℓ, g, h, L, G, and H involved in the solution of $\widetilde{K}_0$. With these variables, the functions required for computing r, θ, and φ read

$$\widetilde{J} = J\left(\frac{3}{2}\,\frac{H^2}{G^2} - \frac{1}{2}\right), \quad R = \sqrt{1 - \frac{12\mu^2\widetilde{J}}{G^4}},$$

$$\frac{\partial R}{\partial G} = \frac{12\mu^2 J}{R\,G^5}\left(\frac{9}{2}\,\frac{H^2}{G^2} - 1\right), \quad \frac{\partial R}{\partial H} = -\frac{18\mu^2 JH}{R\,G^6},$$

$$\widetilde{\sigma} = G\sqrt{\frac{1+2R}{3}}, \quad \widetilde{\mu} = \frac{2}{3}\,\mu\left(\frac{1+2R}{1+R}\right),$$

$$\frac{\partial\widetilde{\sigma}}{\partial G} = \left(\frac{1+2R}{3} + \frac{G}{3}\,\frac{\partial R}{\partial G}\right)\sqrt{\frac{3}{1+2R}},$$

$$\frac{\partial\widetilde{\sigma}}{\partial H} = \frac{G}{3}\,\frac{\partial R}{\partial H}\sqrt{\frac{3}{1+2R}},$$

$$\frac{\partial\widetilde{\mu}}{\partial G} = \frac{2}{3}\,\frac{\partial R}{\partial G}\,\frac{\mu}{(1+R)^2}, \quad \frac{\partial\widetilde{\mu}}{\partial H} = \frac{2}{3}\,\frac{\partial R}{\partial H}\,\frac{\mu}{(1+R)^2},$$

$$a = \frac{\widetilde{\mu}}{\mu^2}\,L^2, \quad e = \sqrt{1 - \frac{\mu^2}{\widetilde{\mu}^2}\,\frac{\widetilde{\sigma}^2}{L^2}}, \quad \sin i = \sqrt{1 - \frac{H^2}{G^2}},$$

$$\cos i = \frac{H}{G}, \quad u - e\sin u = \frac{\mu}{\widetilde{\mu}}\,\ell, \quad 0 \le u < 2\pi,$$

$$\sin v = \frac{\sqrt{1-e^2}\,\sin u}{1 - e\cos u}, \quad \cos v = \frac{\cos u - e}{1 - e\cos u}, \quad 0 \le v < 2\pi,$$

$$\widetilde{v} = v + \left(\frac{\partial\widetilde{\sigma}}{\partial G} - 1\right)(v - u) + \left(\frac{\partial\widetilde{\sigma}}{\partial G} - 1 - \frac{L}{\mu}\,\frac{\partial\widetilde{\mu}}{\partial G}\right)e\sin u,$$

$$\widetilde{\omega} = g + \left(\frac{\partial\widetilde{\sigma}}{\partial G} - 1 - \frac{L}{\mu}\,\frac{\partial\widetilde{\mu}}{\partial G}\right)\frac{\mu}{\widetilde{\mu}}\,\ell,$$

$$\widetilde{\theta} = \theta - \frac{\partial\widetilde{\sigma}}{\partial H}(v - u) - \left(\frac{\partial\widetilde{\sigma}}{\partial H} - \frac{L}{\mu}\,\frac{\partial\widetilde{\mu}}{\partial H}\right)e\sin u,$$

$$\widetilde{\Omega} = h + \left(\frac{\partial\widetilde{\sigma}}{\partial H} - \frac{L}{\mu}\,\frac{\partial\widetilde{\mu}}{\partial H}\right)\frac{\mu}{\widetilde{\mu}}\,\ell.$$

The main difference with the classical case starting from K_0 is that the expansion of K involves the new functions $\widetilde{\sigma}, \widetilde{\mu}$, and their derivatives with respect to G and H. They only depend on these canonical variables but have rather simple expressions using G, H, and R. For automatic algebraic computations (Moons 1991), the non-canonical variable $S = \sqrt{R}$ can be added to the set of variables used to expand the Hamiltonian, provided some minor modifications are made in the derivative algorithms. Then, for the commonly low values of J, the new functions can be quickly expanded in powers of $\left(1 - S^2\right)/S^2$. For very small values of e or i, non-singular canonical variables can be defined in the very same way they are defined for the classical case.

8. Conclusion

The approximation $\widetilde{K}_0$ provides a simple precessing ellipse that is easy to use as a first intermediary orbit in the Hamiltonian context of the J_2 problem, allowing the powerful process of Lie series expansions to compute the complete solution. Suitable for satellites with moderate eccentricities and inclinations, it can be used in the theories of many natural satellites or for non-geosynchronous artificial satellites near the Earth's equator.

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